

Hamiltonian paths on directed grids

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Abstract. Our studies are related to a special class of FASS-curves, which can be described in a node-rewriting Lindenmayer-system. These ortho-tile (or diagonal) type recursive curves inducing Hamiltonian paths. We define a special directed graph on a rectangular grid, and we enumerate all Hamiltonian paths on this graph. Our formulas are strongly related to both the Fibonacci numbers and the domino tilings of chessboards. The constructability of the regular 17-gon with straightedge and compass is also related.

Introduction. In 1877, G. Cantor proved for any positive integer d that, there exists a one-to-one point-to-point correspondence between a unit line segment and the entire d -dimensional space. In other words, the infinite number of points in a unit interval is the same cardinality as the infinite number of points in any finite-dimensional manifold. In 1890, G. Peano constructed a continuous mapping from the unit interval onto the unit square. This continuous curve that passes through every point of the unit square was the first example of space filling curves.

In 1891, D. Hilbert discovered another type of these recursive curves [Sa94]. These FASS-curves (space-**F**illing, self-**A**voiding, **S**imple, self-**S**imilar) can be described in a node-rewriting Lindenmayer-system [PL90,PLF91]. The *ortho-tile* type of FASS-curves can be represented only on a special directed grid graph. (See Figures 8.a and 8.b.) Their approximations are Hamiltonian paths between diagonally located points.

Definitions. Given fixed positive integers p, q , by a *directed grid graph with an odd-even direction*, or by $DGG_{p,q}$, in short, we mean a directed graph on the vertex set $\{1, 2, \dots, p\} \times \{1, 2, \dots, q\}$, as a subset of the xy coordinate plane, with all possible arcs of the form $(x; y) \rightarrow (x+1; y)$ if y is odd, and arcs of the form $(x; y) \rightarrow (x-1; y)$ if y is even, and arcs of the form $(x; y) \rightarrow (x; y+1)$ if x is odd, and arcs of the form $(x; y) \rightarrow (x; y-1)$ if x is even. This graph has pq vertices and $(p-1)q + p(q-1) = 2pq - p - q$ arcs. A *Hamiltonian path* is such a permutation $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{pq}$ of all vertices for which $v_n \rightarrow v_{n+1}$ is an arc for each $n \in \{1, 2, \dots, pq-1\}$. Let $h(p, q)$ denote the number of Hamiltonian paths in $DGG_{p,q}$. Furthermore, for a positive integer r , let $h_r(p, q)$ denote the number of those Hamiltonian paths of the form $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{pq}$ for which $v_1 = (1; 1)$, $v_2 = (2; 1)$, ..., $v_r = (r; 1)$ hold but $v_{r+1} \neq (r+1; 1)$. For other definitions and for the history of the Hamiltonian paths we refer reader to [Ba06] and [We01].

In Figure 1 we show $DGG_{5,4}$, and in Figures 2.a and 2.b we give two Hamiltonian paths.

Basic observations. We can make the following easy observations: For any positive integers p, q , we have $h(p, q) = h(q, p)$ and $h(p, 1) = h(1, q) = 1$. If p is odd, then $h(p, 2) = h(2, p) = 1$. If both p and q are even, then $h(p, q) = 0$. If pq is odd, then $h(p, q) > 0$.

We can also observe that for any Hamiltonian path $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{pq}$ we have $v_1 = (1; 1)$. On the other hand, if both p and q are odd, then $v_{pq} = (p; q)$. If p is odd but q is even, then $v_{pq} = (1; q)$. Finally, if p is even but q is odd, then

$v_{pq} = (p; 1)$. For example, in case of the Hamiltonian paths of Figures 2.a and 2.b, the terminal points are in the top-left corner.

We can also make the following observations: If r is even, then $h_r(p, q) = 0$. Therefore

$$h(p, q) = h_1(p, q) + h_3(p, q) + h_5(p, q) + \cdots + h_{p-1}(p, q)$$

if p is odd, and

$$h(p, q) = h_1(p, q) + h_3(p, q) + h_5(p, q) + \cdots + h_{p-1}(p, q)$$

if p is even. If both p and q are odd and both are at least 3, then $h_1(p, q) > 0$, $h_3(p, q) > 0$, ..., $h_p(p, q) > 0$. If p is odd and $q \geq 2$, then $h_p(p, q) = h(p, q - 1)$. If $q \geq 3$, then $h_1(3, q) = h(3, q - 2)$.

Fibonacci numbers. Among the above observations we find that $h(1, 3) = h(3, 1) = 1$, $h(2, 3) = h(3, 2) = 1$, and for $n = 3, 4, 5, \dots$, we have

$$h(3, n) = h_1(3, n) + h_3(3, n) = h(3, n - 2) + h(3, n - 1).$$

Therefore the numbers $h(1, 3) = h(3, 1)$, $h(2, 3) = h(3, 2)$, $h(3, 3)$, ... are exactly the well-known Fibonacci numbers: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, ..., $F_n = F_{n-1} + F_{n-2}$, Up to now we gained the following table of the values $h(p, q)$ for $\min\{p, q\} \leq 6$.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$h(p, 1) =$	1	1	1	1	1	1
$h(p, 2) =$	1	0	1	0	1	0
$h(p, 3) =$	1	1	2	3	5	8
$h(p, 4) =$	1	0	3	0		0
$h(p, 5) =$	1	1	5			
$h(p, 6) =$	1	0	8	0		0

In the rest of the present paper we focus on the cases where $\min\{p, q\} \geq 4$.

Definitions. By a *domino* we mean such a rectangle whose corners are all vertices of our digraph, and in the rectangle the length of the diagonals is exactly $\sqrt{5}$. We say that some pairwise nonoverlapping dominoes form a *domino tiling* of our graph if their union is the entire grid, i.e. the rectangle with corners $(1; 1)$, $(p; 1)$, $(p; q)$, $(1; q)$. Clearly, the number of dominoes in a domino tiling is $(p - 1)(q - 1)/2$. We say that a given Hamiltonian path $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{pq}$ *avoids* a given domino if the shorter symmetry axis of the domino is not an arc of the Hamiltonian path.

Theorem 1. *There is a one-to-one correspondence between Hamiltonian paths and the domino tilings such that for each domino tiling the corresponding Hamiltonian path is the only one which bisects no domino of the tiling.*

We will prove the theorem later. In Figures 3.a and 3.b we find the domino tilings that correspond to the Hamiltonian paths of Figure 2.a and 2.b., respectively.

More than a half a century ago [Ka61] and [Te61] proved that the number of different domino tilings is exactly

$$\prod_m \prod_k \left(4 \cos^2 \frac{m\pi}{p} + 4 \cos^2 \frac{k\pi}{q} \right)$$

where the products are understood for all positive integers m and k such that $2m < p$, $2k < q$. This formula is well-known as the *Kasteleyn* formula.

Example. We consider the case $p = q = 5$. Now

$$4 \cos^2 \frac{\pi}{5} = \frac{3 + \sqrt{5}}{2} \quad 4 \cos^2 \frac{2\pi}{5} = \frac{3 - \sqrt{5}}{2}$$

$$\Pi_m \Pi_k \left(4 \cos^2 \frac{m\pi}{p} + 4 \cos^2 \frac{k\pi}{q} \right) = (3 + \sqrt{5}) \cdot 3^2 \cdot (3 - \sqrt{5}) = 36$$

Therefore, by Theorem 1 we gain that $h(5, 5) = 36$.

Remark. The above mentioned Kasteleyn formula outputs 0 if both p and q are even. If $\min\{p, q\} = 1$, the meaning of the formula is 1. If $\min\{p, q\} = 2$ and $|p - q| \in 1, 3, 5, \dots$, then the Kasteleyn formula outputs 1. Since $4 \cos^2 \frac{\pi}{3} = 1$, if $\min\{p, q\} = 3$, then the Kasteleyn formula and our Theorem 1 produces the following two nice formulas for the Fibonacci numbers F_{2n} and F_{2n+1} for any positive integer n .

$$\prod_{k=1}^{n-1} \left(1 + 4 \cos^2 \frac{k\pi}{2n} \right) = F_{2n} \quad \prod_{k=1}^n \left(1 + 4 \cos^2 \frac{k\pi}{2n+1} \right) = F_{2n+1}$$

For example for $n = 8$ and for $x_k = 1 + 4 \cos^2 \frac{k\pi}{17}$, $k = 1, 2, \dots, 8$, the latter formula produces $x_1 x_2 \cdots x_8 = 1597$. However, by the famous result of Gauss on the constructability of the regular 17-gon with straightedge and compass, we can derive a nice formula for each x_k separately. Namely, we can start from the well-known formula (see, e.g., [Gi07])

$$16 \cos \frac{2\pi}{17} = \sqrt{17} - 1 + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}$$

and we can apply that

$$1 + \cos \frac{2\pi}{17} = 2 \cos^2 \frac{\pi}{17} \quad 1 - \cos \frac{2\pi}{17} = 2 \sin^2 \frac{\pi}{17}$$

This way we can express each x_k as $\frac{1}{16}$ times an integer coefficient polynomial of $v = \sqrt{34 - 2\sqrt{17}}$ and $w = \sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}$ because $64 \cos^2 \frac{\pi}{17} = 64 + (2 - v)v + 4w$. For example $x_1 = 5 + \frac{(2-v)v}{16} + \frac{w}{4}$.

Table of $h(p, q)$. By the Kasteleyn formula and by our Theorem 1 we can continue the above incomplete table as follows

	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$
$h(p, 4) =$	0	11	0	41	0	153
$h(p, 5) =$	11	36	95	281	781	2245
$h(p, 6) =$	0	95	0	1183	0	14824
$h(p, 7) =$	41	281	1183	6728	31529	167089
$h(p, 8) =$	0	781	0	31529	0	1292697
$h(p, 9) =$	153	2245	14824	167089	1292697	12988816

Definition. Given a domino D , there are six vertices at the perimeter of the domino; the *canonical numbering* of these vertices is D_1, D_2, \dots, D_6 if all five $D_j \rightarrow D_{j+1}$ are arcs in the graph, $j = 1, 2, \dots, 5$. In Figure 4 we can see a canonical numbering. Observe, that each domino has exactly one canonical numbering. In this paper we always consider canonical numberings.

The proof of Theorem 1. The cases $\min\{p, q\} \leq 2$ are obvious. In the rest of the proof we assume that $\min\{p, q\} \geq 3$. The case when both p and q are even is also obvious. By symmetry we may assume that p is odd. We have at least one Hamiltonian path and we have at least one domino tiling.

Let us consider a fixed Hamiltonian path H and a fixed domino D . According to the canonical numbering let the vertices around D be D_i , $i = 1, 2, \dots, 6$. Observe that if $D_5 \rightarrow D_2$ is not an arc in H , then $D_5 \rightarrow D_6$ must be an arc in H , and $D_1 \rightarrow D_6$ cannot be an arc in H . Given H and given k pairwise nonoverlapping dominoes, we gain $2k$ arcs such that none of them can be in H . These $2k$ arcs are all distinct because the $D_5 \rightarrow D_2$ arcs are all inside the pairwise nonoverlapping dominoes, and each $D_1 \rightarrow D_6$ arc belongs to the domino situated in the right angle determined by the two arcs started at D_1 . In a domino tiling there are $(p-1)(q-1)/2$ pairwise distinct dominoes. In summary, if H avoids each domino in a fixed domino tiling, then there are $pq-1$ arcs in H and there are $(p-1)(q-1)$ further arcs not in H . However

$$pq-1 + (p-1)(q-1) = 2pq - q - p$$

This is exactly the total number of arcs in the graph. Therefore, starting from the given domino tiling, some (but not necessary all) of the canonical numbering $D_j \rightarrow D_{j+1}$ arcs of the domino's perimeter form the Hamiltonian path. On the other hand, the $(p-1)(q-1)$ arcs missing from the a Hamiltonian path determine two-by-one all dominoes of the domino tiling.

Now we make the above argument more explicit. As on a usual chessboard, we say that a unit square of our grid is black if the sum of bottom-left corner coordinates is even. (See Figure 5 as an illustration.) The other unit squares are called white. Clearly, each domino D consists of a black square and of a white square; the corners of the black square are D_1, D_2, D_5, D_6 according to the canonical ordering, and the white square's corners are D_2, D_3, D_4, D_5 . If the domino is an element of the domino tiling, then the arc between the domino's squares must not be in the corresponding Hamiltonian path. These $(p-1)(q-1)/2$ pairwise distinct arcs of $DGG_{p,q}$ are called *domino-axis* arcs. According to the canonical ordering of a domino D , the domino-axis arc is the $D_5 \rightarrow D_2$ arc. On the other hand, if a Hamiltonian path avoids a domino D , then the $D_1 \rightarrow D_6$ arc can not be in the Hamiltonian path, either. We call these arcs *domino-black-end* arcs. Clearly, each domino-axis arc corresponds to exactly one domino in a domino tiling, and each domino-black-end arc corresponds to exactly one domino in a domino tiling, too. The total number of domino-axis arcs and domino-black-end arcs is $(p-1)(q-1)$. The number of the remaining arcs is exactly the same as the number of the arcs in a Hamiltonian path. Therefore a domino tiling determines at most one Hamiltonian path which bisects no domino of the tiling.

We can observe that in case of a domino D of a domino tiling, the arcs $D_1 \rightarrow D_2$ and $D_5 \rightarrow D_6$ can be neither domino-axis arcs nor domino-black-end arcs. Therefore both arcs must be in the Hamiltonian path corresponding to the domino tiling.

Now let us consider such arcs of the grid graph which are at the perimeter of the grid and which are at the perimeter of a white square at the same time. One can easily compute that there are $(p-1)/2$ such arcs at the bottom line, $(p-1)/2$ such arcs at the top line, and there are other $q-1$ such arcs at the vertical sides of the grid. We call such arcs as *white perimeter* arcs. Observe that each white perimeter arc must be in each Hamiltonian path. Obviously, a perimeter arc can

be neither a domino-axis nor a domino-black-end arc in case of any domino tiling. The total number of white perimeter arcs is $p + q - 2$. Since a Hamiltonian path contains $pq - 1$ arcs, there are exactly $pq - 1 - (p + q - 2) = (p - 1)(q - 1)$ arcs in any Hamiltonian path which are neither white perimeter arcs, nor domino-axis arcs nor domino-black-end arcs. Given a domino tiling, we call all arcs as *domino-black-side* arcs which are neither white perimeter arcs, nor domino-axis arcs nor domino-black-end arcs.

Observe that around any black square, any Hamiltonian path contains exactly two opposite arcs. Therefore from all black squares a Hamiltonian path contains exactly $(p - 1)(q - 1)$ arcs.

Let A be the set of all arcs in the grid which are not white perimeter edges. We have that

$$|A| = 2pq - q - p - (p + q - 2) = 2(p - 1)(q - 1)$$

(In Figure 6 we find an illustration for $p = 4$, $q = 3$.) Each Hamiltonian path contains exactly half of the arcs in A .

Each domino tiling also takes exactly a quarter of the arcs of A as domino-black-end arcs and exactly a quarter of the arcs as domino-axis arcs. And these three parts must be pairwise disjoint if and only if the Hamiltonian path bisects neither domino of the tiling.

We obtained that a domino tiling determines the corresponding Hamiltonian path. On the other hand, any Hamiltonian path determines for each black square that in a domino tiling the domino containing the black square is in vertical position or in horizontal position. This leaves one or two choices for the white neighbor. However, if there are two choices, then both are vertical or both are horizontal. In case of the bottom-left black square there is only one choice. (In Figures 7.a and 7.b we find illustrations for $p = 9$, $q = 9$.)

Given a Hamiltonian path H we make a bipartite graph B_H as follows: The black squares form one color class of the vertices in B_H and the white squares form the other color class of the vertices. A black and a white square will be adjacent in B_H if they are neighbors and the two squares form such a domino whose position is allowed in the previous sense. Observe that the Hamiltonian path allows no cycle in B_H . Therefore, there exists at most one perfect matching in the bipartite graph. Therefore a given Hamiltonian path H allows at most one domino tiling whose dominoes are all avoided by H . This completes the proof of Theorem 1.

References

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Figures

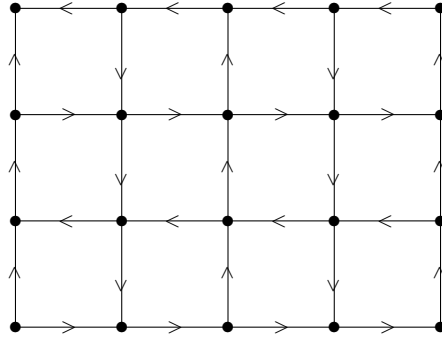


Figure 1. The directed graph $DGG_{5,4}$

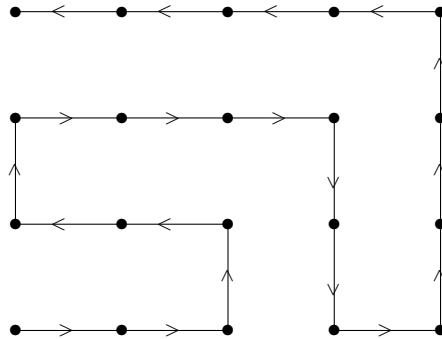


Figure 2.a. A Hamiltonian path

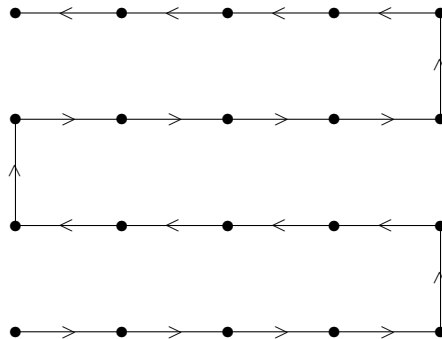


Figure 2.b. Another Hamiltonian path

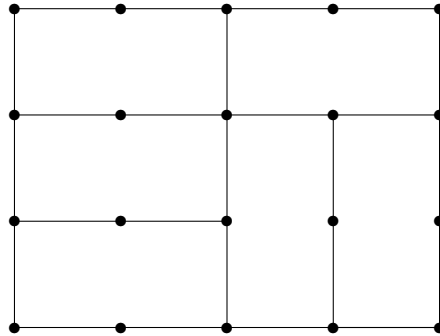


Figure 3.a. A tiling corresponding to Fig. 2.a

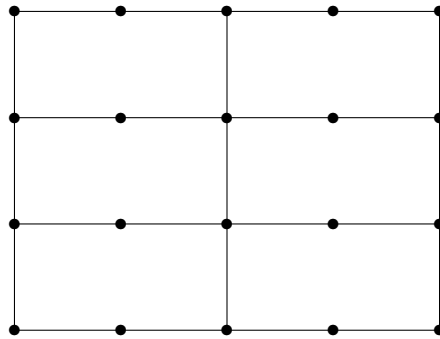


Figure 3.b. A tiling corresponding to Fig. 2.b

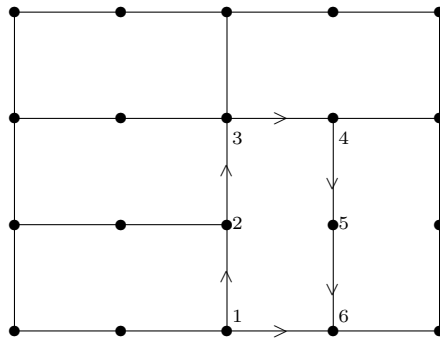


Figure 4. A canonical numbering around a domino

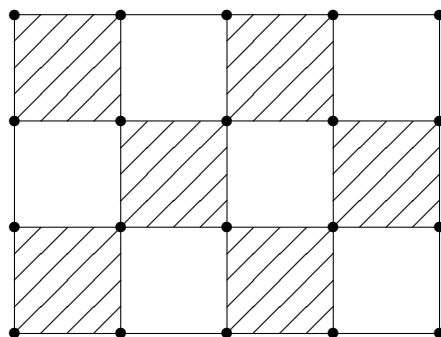


Figure 5. The grid as a chessboard

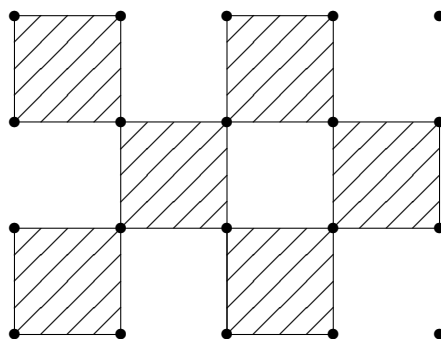


Figure 6. Without the white perimeter arcs

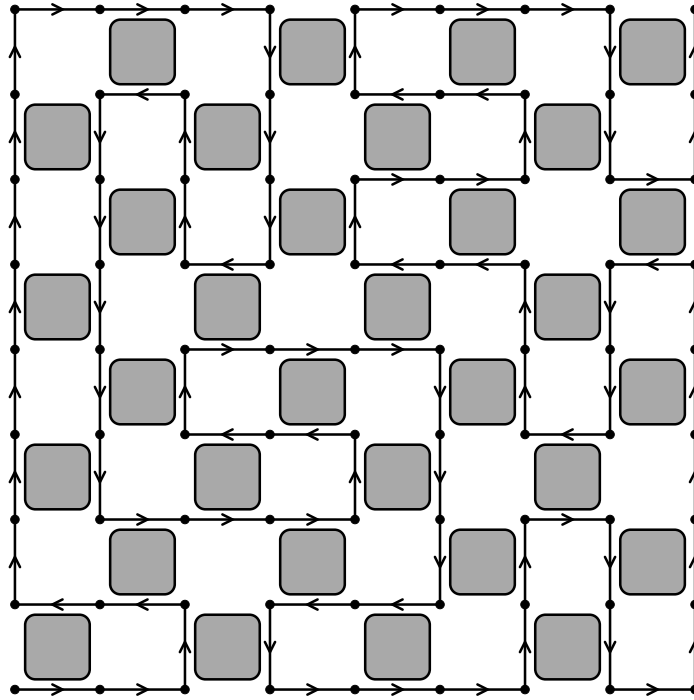


Figure 7.a. The opposite arcs of the black squares and the white perimeter arcs form a Hamiltonian path covering a Euclidean disk.

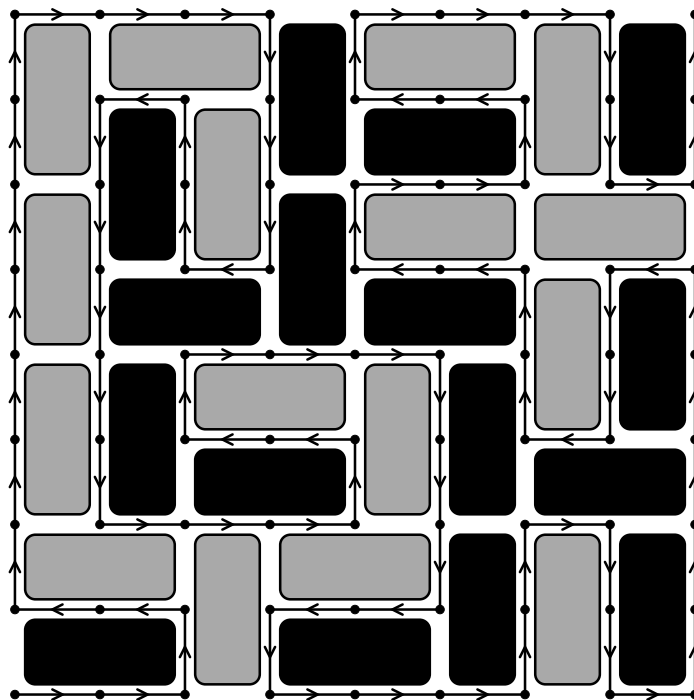


Figure 7.b. A Hamiltonian path and the corresponding domino tiling.

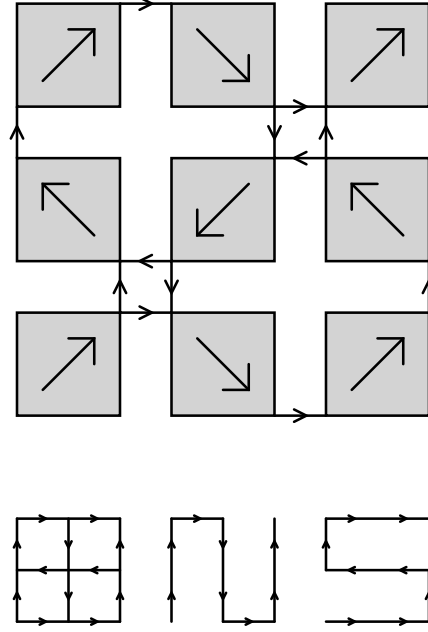


Figure 8.a. Possible connecting edges between self similar parts
in ortho-tile type, node-rewriting FASS-curves.
(Grey squares grow out from the nodes of the path.)

Figure 8.b. Directed Grid Graph: $DGG_{3,3}$
and the 2 possible Hamiltonian paths on it.